Development of a computational model, to analyse the motion of bodies in the solar system, with respect to launching an extra solar satellite, by performing a gravity assist on the planet Jupiter

Aaron Jones[1]
3rd Year MSci Physics Undergraduate, School of Physics and Astronomy, University of Birmingham[a)]

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This report address the issue of launching an extra solar satellite from earth and doing so by achieving a gravity assist from the planet Jupiter. A theoretical calculation finds that, by using a combination of a gravity assist and the velocity of the Earth, it is possible to achieve a significant reduction in launch energy. This reduction equates to an approximately 50% reduction in launch velocity. A multi body gravitational interaction model is produced and used to solve the gravitational interactions between: the Sun, the satellite, Earth, Jupiter and Mars. This model is then used to test the theory. It is found that gravity assists limit the direction and time of launch of the satellite significantly.

Keywords: Computational Modelling, Satellites, Solar, Orbits, Gravity Assist

I. INTRODUCTION

There is scope in modern day telecommunications and scientific research for a requirement to produce satellites that may need to travel beyond the inner solar system. Several such satellites have already been built and tested, notably, Voyager 1 & 2. These satellites were launched on Sept 5 1977 and they have travelled beyond the solar system. This report addresses two key issues. Firstly, to present a suitable model that can be used for predicting the motion of the satellite throughout the solar system, hence solving a multi-body gravitational problem. Secondly, this report discusses the advantages and challenges of using a gravitational assist, around the planet Jupiter, to reduce the launch payload. Finally, an attempt will be made to optimise the trajectory, however, a full solution is beyond the scope of this project and is no trivial task[1].

A. Gravitational Assists

It is possible for a satellite to perform a gravitational assist (slingshot) off a large body, such as a planet, and hence gain an increase in its energy. As the satellite travels though the gravitational field of the lager body, its velocity may be increased substantially. The collision is elastic and so the planet will slow down, however, since the mass is large, the effect will be small. This effect is somewhat analogous to bouncing a ping pong ball off a train.

1. Proof

In the centre of mass frame, if the satellite has enough energy to escape the gravitational interaction with the planet, the satellite will enter a hyperbolic orbit, given by the equation[2],

$$r = \frac{1}{\frac{1}{2}(\frac{1}{r_0} + \frac{1}{r_1} - (\frac{1}{r_1} - \frac{1}{r_0})\cos\theta)}. \tag{1}$$

Where r is the distance between the two bodies, r_1 and r_0 are known orbital parameters. This can be plotted using the code available in Appendix C. Since the mass of the planet is much greater than the mass of the satellite, $m_p >> m_s$, the centre of mass (COM) can be approximated to be at the centre of the planet. Therefore, the apparent velocity of the satellite, parallel to the planet's motion, will be the difference between the two velocities, $v_{s||} - v_p$. In this COM frame, the satellite will complete a hyperbolic orbit around the planet. Since energy is conserved, it is well known that the satellite's incoming and outgoing velocities must be equal in magnitude. Therefore, the satellite will leave the frame with COM velocity, $-(v_{s||} - v_p)$, exactly the same as when it entered the frame. Therefore, in the solar frame the satellite will have velocity,

$$-(v_{s||} - v_p) + v_p = -v_{s||} + 2v_p. \tag{2}$$

If the satellite and the planet have velocities that are initially opposed to each other then the satellite will net gain twice the speed of the planet in the solar frame. If the planet and the satellite have velocities that are initially in the same direction then the satellite will net lose twice the speed of the planet.

2. Non-Head On Collisions

A head on collision will gain $2v_p$, as described above; however, a non-head on collision will not get the full gain of the interaction with the planet. The motion can be broken down into the motion for the x and y components, where the planet moves only in the x axis. Since the collision is elastic and $m_p >> m_s$, the y velocity of the

a) Electronic mail: axj120@bham.ac.uk

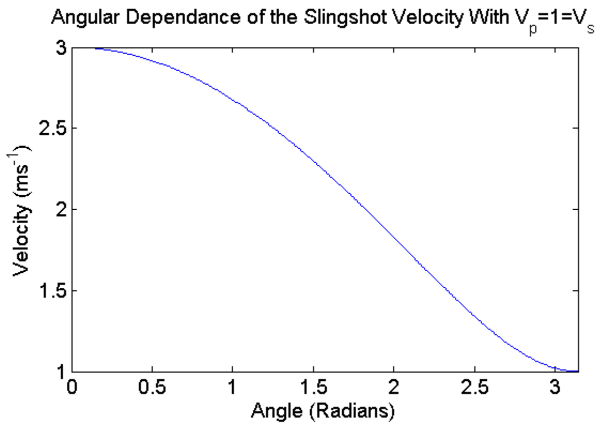

FIG. 1. This graph shows the dependence of the velocity gain as a result of the gravity assist, on the angle between the planet's velocity and the satellite's approach velocity.

planet will not change during the interaction. Therefore, $v_{ys} = v'_{ys} = v \sin\theta$ where v and v' are velocity before and after the interaction respectively. θ is the angle of approach. Thus, $v_x = v\cos\theta$ and,

$$v'_x = v_x + 2v_p = v_s \cos\theta + 2v_p. \quad (3)$$

Thus combining v'_{sx} and v'_{sy},

$$v'_s = \sqrt{(v_s \cos\theta + 2v_p)^2 + v_s^2 \sin^2\theta} \quad (4)$$

$$= \sqrt{(v_s^2 + 4v_p v_s \cos\theta + 4v_p^2}. \quad (5)$$

This has linier dependence on v_s and v_p and a cosine dependence the angle as shown in figure 1. If this slingshot is required to overcome further gravitational potential energy then the difference in the energy will be of interest. Since,

$$\Delta E = \frac{1}{2} m \Delta(v^2), \quad (6)$$

equation 4 can be re-arranged and substituted into 6 to give,

$$\Delta E = \frac{1}{2} m(4v_p v_s \cos\theta + 4v_p^2). \quad (7)$$

Of course, for this to be notable there is an implicit assumption that the gravitational potential must be, in some way, comparable to the kinetic energy of the satellite.

B. Minimum Approach Velocity

If the satellite approaches Jupiter, with a velocity in excess of the escape velocity, $(60500ms^{-1})$, then according to the theory, the satellite will gain $2v_p$. However, if the satellite does not have this velocity, then it may gain the escape velocity from the planet. Thus a satellite must approach the planet with a velocity, $(v_{s||} - v_p)^2 + v_{s\perp}^2 > v_{esc}^2$ in order to escape the planets gravitational field and continue its journey. It is clear here that this will introduce an additional angular effect into the trajectory. This is an area that with more time, could be researched further. Finally, it is known that,

$$v_{esc} = \sqrt{2Gm_p(\frac{1}{r_1} - \frac{1}{\infty})}, \quad (8)$$

and if we assume that the satellite approaches from infinity, then $r_1 = \infty$ and $v_{esc} = 0$. So this effect should not be present in the real system.

C. Calculating the Best Possible Trajectory

Given the theory discussed above, there are presently two basic trajectory options. Firstly, using the velocity of the earth to provide some energy to get to Jupiter. However, since all of the planets orbit the sun in the same direction, the best possible approach angle will be $\theta > \frac{\pi}{2}$. Thus achieving an maximum energy gain of,

$$\Delta E = \frac{1}{2} m(4v_p^2). \quad (9)$$

This will be referred to as a half gravity assist. Alternatively, the satellite could be launched in anti-phase with the earth. Initially expending more energy, but achieving an energy gain of up to,

$$\Delta E = \frac{1}{2} m(4v_p v_s + 4v_p^2), \quad (10)$$

from the interaction with Jupiter. Referred to herein as a full gravity assist.

1. Half Gravity Assist

The escape energy of the solar system is given by,

$$E = Gm_s M_{sol} \times \left(\frac{1}{r_1} - \frac{1}{\infty}\right), \quad (11)$$

where, m_{sol} is the mass of the solar system, m_s is the mass of the satellite; this will be set to $117Kgs$ as this was the mass of the Voyager craft. All other parameters have their usual meaning. The escape energy from Jupiter, $r = r_j = 5.195Au$, is $1.907 \times 10^{32}J$ and conveniently equals the energy gained from a half gravity assist manoeuvre to 3 significant figures. Since some energy is being provided by the motion of the earth, the launch energy (i.e. the energy required once outside the earth gravitational field) will be given by,

$$E = Gm_s M_{sol} \times \left(\frac{1}{r_e} - \frac{1}{r_j}\right) - \frac{1}{2} m_s v_e^2 \quad (12)$$

$$= 2.81 \times 10^{10} J. \quad (13)$$

2. Full Gravity Assist

For a full gravity assist $4.23 \times 10^{10} J$ would be obtained from the gravitational interaction with Jupiter. This would be more than sufficient to escape the solar system and would result in a large velocity once out of it. Therefore, the only energy required is that to, a) overcome the earths initial velocity and, b) get to Jupiter. Hence the required launch energy (i.e. the energy required once outside the earth gravitational field) will be given by

$$E = Gm_s M_{sol} \times \left(\frac{1}{r_e} - \frac{1}{r_j}\right) + \frac{1}{2} m_s v_e^2, \quad (14)$$

$$= 1.32 \times 10^{11} J. \quad (15)$$

This is 5 times more than that required for a half gravity assist. Some of this energy could of course be gained by slowing the satellite down using a slingshot off Venus.

3. Conclusions

For a simplistic 4 body problem, the optimum route choice is clearly a half slingshot off Jupiter. However, for more complicated missions, if a high final velocity is a requirement, a more complicated route, involving slowing the satellite down and achieving a reverse full gravity assist off Venus, may prove to be a viable option. However, more time would be needed to calculate this. One such situation would be human interstellar travel, when the additional launch energy and time spent around Venus may save a significant amount of journey time.

II. THE MODEL

To analyse the motion of the satellite during a gravity assist, a three body problem will need to be constructed containing the Sun, the satellite and Jupiter. This is impossible to solve analytically, however, it is possible to solve computationally. At this point, the value in pursuing analytical solutions diminishes and it becomes more sensible to construct a n body solution, using solely Newtons laws of motion,

$$\mathbf{F} = m(a). \quad (16)$$

Where m is the mass (as before) and $\mathbf{a}$ is the acceleration. $\mathbf{F}$ is given by,

$$\mathbf{F_i} = \frac{Gm_i m_j \mathbf{r_{i \to j}}}{r^3}. \quad (17)$$

Where r is a vector from body i to body j and all other parameters have their usual meaning.

A. The Two Body Model

For two bodies, the force can be evaluated using equation 17. Using equation 16, an acceleration can be found for each body. A second order Runge Kutta integration routine can then be utilised to obtain the positions of both bodies as a function of time. By using the conservation of angular momentum, an approximate error on each numerical step can be calculated. Thus an adaptive step size can be implemented, to only allow steps where the change in the total angular momentum does not exceed 0.1% of the total value.

B. Extension to n bodies

It is possible to extend this to n bodies by using a pair of nested for loops. This will analyse the forces between each pair of bodies individually. Then, by summing the results to find an overall acceleration and utilising a Runge Kutta routine to evolve the system, a time dependant model can be generated for n bodies. This is computationally intensive however, since the only assumption is that equation 16 holds true, it can be used to check the assumptions made in the previous calculations. It will also highlight any unexpected results of the slingshot. Although it is recommended to re-run the routine using a higher accuracy (smaller step size) before researching unusual results.

III. RESULTS

The model was produced and three sets of results were obtained. However, for the fine details of the slingshot, step sizes of $10^{-300}s$ were required. As such, a minimum step size of $0.1s$ was implemented.

A. Test 1 : Angular Dependence on Trajectory

In Section I A 2 a cosine dependence on energy gained and launch angle was predicted. By setting up the model appropriately, this dependence was tested.

1. Method

Jupiter was placed at the origin, $(x, y) = (0, 0)$, with velocity component completely in the x direction. The satellite was placed a fixed radius, r, away and aimed towards the origin, while the angle, θ, was varied. Measures were taken to detect a crash and no crashes occurred. The velocity of the satellite was $1000ms^{-1}$, the only bodies in the integration routine were the satellite and Jupiter.

2. Results

The theory predicted results with an angular cosine dependence and energy gain of around $10^{10}J$. The results obtained matched this prediction; however, the angular dependence was found to be much stronger than expected, whilst the energy gain was slightly less than expected. The energy gain can be put down to insufficiently small step sizes. The cause of the stronger angular dependence is unknown. One possible cause is that the initial velocity was slightly less than the escape velocity, so the effects mentioned in Section I B may appear. Further details and results are available in Appendix B.

B. Test 2 : Achieving a Gravitational Assist From Jupiter

It was possible, using the model, to attempt to repeat a journey, based on that of Voyager 1, though the solar system and achieve a gravity assist off Jupiter. This was achieved with fairly successful results.

1. Method

To achieve this assist, the model was set up with 5 celestial bodies: the Sun, placed at $x = 1, y = 0$ with nil velocity; the Earth, Mars and Jupiter, all placed with their approximate position and velocity on the launch date, and finally a satellite. The satellite was refered to as Voyager, due to its similarity to the missions. This weighed $117Kg$ and was placed $1km$ above the Earths surface, with velocity radially outwards of $\approx 1000ms^{-1}$, at midnight local time. Mars was not used, however, since the satellite passes close by, it was included for accuracy.

2. Observations

Two key observations of the model were made during this analysis:

a. Errors in the System The non-adaptive integration routine, with 3 hour step sizes, would reach Jupiter with a launch velocity as low as $8080ms^{-1}$. This is much less than the predicted $2191ms^{-1}$ in section I C 1. However, using the adaptive routine, a launch velocity of half the predicted amount would reach Jupiter at its aphelion. This suggests that the shorter step sizes give a more accurate representation of the system. It also suggests that the model has a tendency to underestimate the strength of the gravitational interactions.

b. Optimum Trajectory It was noted that if the satellite performs a half gravity assist off Jupiter and is launched with precisely the amount of energy to reach Jupiter, it will be at its aphelion at Jupiter and hence be travelling in the same direction. Thus it will 'reverse' gravity assist, hence losing energy in the solar frame. Therefore the satellite must be launched with significantly more velocity than that required to reach Jupiter and hence when it does it will be travelling close to perpendicular to the planet's motion and will perform a gravity assist. An example would be;

```
t = t - year2sec(0.25) + (3600*24*30) +
    (3600*24*16.5); //Launch 0.25yr earlier and
    1 month later and 16.5 days later than
    original launch date,
double velin = 2.1921e+04 - 0.5*2.1921e+04;
    //Predicted launch velocity - adjustment
    for large steps
```

providing a poor slingshot and launching with twice the velocity,

```
t = t - (3600*24*8.125);
double velin = 2.1921e+04; //Predicted launch
    velocity
```

will achieve a much better slingshot since the approach angle is closer to $\pi/2$.

C. Test 3 : Launch Sensitivity

The launch dates used for the model Voyager mission, discussed in section III B above, was found to heavily affect the power of the achieved slingshot. Whilst the satellite always seemed to exceed solar escape velocity (although, this may be due to errors in the later step sizes,) the amount by which it exceeded this varied by almost 100%, in the space of 2 hours.

1. Method

The model used in section III B was used and the following line was adjusted to control the launch date:

```
t = t - year2sec(0.25) + (3600*24*30) +
    (3600*24*16.5); //Launch 0.25yr earlier and
    1 month later and 16.5 days later than
    original launch date
```

Data was outputted after every 2 hours of simulated time and the integration ran for 2 years. The velocity line was set to:

```
double velin = 2.1921e+04; //Predicted launch
    velocity - adjustment for large steps
```

2. Results

The results clearly show that over a range of a few hours, the satellite could miss Jupiter or gravity assist in either direction. Thus launch dates must be accurate to within a few hours. Since the Earth rotates on its axis, this limits the launch site to somewhere where it will be midnight local time, at the UTC launch time given,

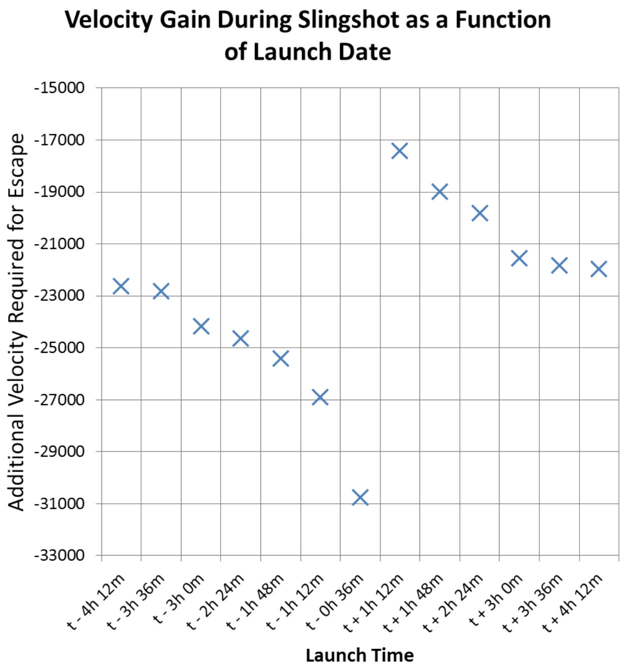

FIG. 2. This graph shows how the difference between the final velocity of the satellite and the escape velocity of the satellite varies, as launch time is changed. It should be noted that the rotation of the earth is not accounted for. A negative number indicates the final velocity is in excess of the escape velocity at that point. t_0 is set as the time at which a launch would pass dangerously close to Jupiter and crash.

to result in the best trajectory. In theory this limits the launch location, however, in practise launches are complex and the rocket itself will be able to make minor course adjustments. The results are illustrated in figure 2

IV. CONCLUSIONS

A number of final conclusions can be made. Firstly, it can be concluded that it is possible to use a large solar bodys velocity, to gain energy in the solar frame via the gravity assist mechansim discussed above. It can also be concluded that for most situations, a half gravity assist is the preferable in terms of saving launch energy. Thirdly, it is possible to approximately solve the solar n body problem using modern computing techniques. One method of doing this is presented with this report and used for this analysis. Using this model, it was determined that an approximately 50% reduction in launch velocity can be achieved by performing a Jovian gravity assist. Although, this approach has a 2 key limitations: firstly, the launch must be at a very precise time and secondly, this limits the direction that the satellite can be launched in. A final remark should be made, noting that a bug was found in the adaptive step size routine. This was corrected, but the data was not updated due to time constraints.

A. Potential For Further Investigation

There are two possible routes for further investigation. A reduction in step sizes is likely to be difficult to achieve, however, it is very easy to add more bodies to the integration routine. Hence, one possible route for further investigation is to explore more complex paths though the solar system. An alternative route would be to adjust the program to allow the satellite to analyse its own trajectory and choose its route, so on each step the satellite would work out a correction to its course.

Appendix A: Acknowledgement & Bibliography

The author would like to thank the authors of Wolfram|Alpha, all initial data such as constants, masses & radii has been obtained from this source[5]. Critical data was cross checked against the NASA factsheets[6],[7] &[8] and the solar radius was obtained from recent research by Marcelo Emilio et al[9].
The initial planetary positions are approximate and were obtained by analysing the image produced by site "The Planets Today"[11]. This site gets its data from the NASA Horizons web interface[12].
The author would also like to thank the members of the Stack Overflow community for their answers to some syntax related questions[3] and the University of Birmingham webpages on $C++$ programming[4] for their help.
This document was prepared using LaTeX 2_ε and the ShareLatex AIP templates[10]. The author would like to thank the authors of the GSL library for the Runge Kutta 4 routine used in this software.

REFERENCES

[1] Carnelli, I. Dachwald, B. and Vasile, M. (2007) [online]. Optimizing low-thrust gravity assist interplanetary trajectories using evolutionary neurocontrollers. In: Evolutionary Computation, 25-28 Sept. 2007, Singapore. pp. 965 - 972. Available from: <http://ieeexplore.ieee.org/xpl/articleDetails.jsp?arnumber=4424574>. [Accessed 07 Jan 2013].
[2] Prof. A. Freise (2013). Lecture notes distributed in the topic 03 21286 Computational Modelling of Physical Systems, Mathematics 27: The Two (and Three) Body Problem(s). School of Physics and Astronomy, University Of Birmingham. 2013/2014 ed.
[3] Stack Overflow (2013). Forum [online]. - Available from: <http://stackoverflow.com/>. [Last Accessed 07 Jan 2013]
[4] Hillier, S. (2013). University of Birmingham - Introduction to C [online]. Available from: <http://epweb2.ph.bham.ac.uk/user/hillier/course3cpp/>. [Accessed 22 Dec. 2013].
[5] Wolfram|Alpha (2013). Encyclopaedia [online]. Available from: <https://www.wolframalpha.com/>. [Last Accessed 07 Jan 2013].

[6]Dr. David R. Williams (2013). Earth Fact Sheet [online]. Available from: <http://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html>. [Accessed 07 Jan 2013].
[7]Dr. David R. Williams (2013). Sun Fact Sheet [online]. Available from: <http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.htmll>. [Accessed 07 Jan 2013].
[8]Dr. David R. Williams (2013). Jupiter Fact Sheet [online]. Available from: <http://nssdc.gsfc.nasa.gov/planetary/factsheet/jupiterfact.html>. [Accessed 07 Jan 2013].
[9]Marcelo Emilio, Jeff R. Kuhn, Rock I. Bush, Isabelle F. Scholl (2012). Measuring the Solar Radius from Space during the 2003 and 2006 Mercury Transits. The Astrophysical Journal [online]. 750, p.135. Available from: <http://iopscience.iop.org/0004-637X/750/2/135/article >(limited access) or <http://arxiv.org/abs/1203.4898>(free access). [Accessed 07 January 2013].
[10]Share Latex [online]. (2013). Available from: <https://www.sharelatex.com/>. [Accessed 17 January 2014].
[11]The Planets Today [online]. (2013). Available from: <http://www.theplanetstoday.com/>. [Accessed 09 December 2013].
[12]NASA HORIZONS Web-Interface [online]. (2013). Available from: <http://ssd.jpl.nasa.gov/horizons.cgi>. [Accessed 07 January 2014].

Appendix B: Test 1 : Angular Dependence on Trajectory

One of the tests performed using the model was to test the angular dependence of the incoming satellite on the energy gained by it against formula 7. This test was completed by setting up the integration routine to model two bodies. Jupiter was placed at $(0, 0)$, the satellite was then placed at $(r\cos\theta, r\sin\theta)$ where $r = 20r_j$ and $\theta = [\pi/6, \pi/4, 3\pi/6, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6]$. The launch velocity was $1000ms^{-1}$ and projected towards the centre of Jupiter. Since Jupiter was moving, the satellite never crashed into the planets surface. A minimum step size of $0.1s$ was enforced and every time the model ran, this step size would have been exceeded. From this it is known that the error on the total angular momentum was in excess of 0.1% on every step. Nonetheless, some results were obtained and they matched the predictions in both their order of magnitude and their shape.

TABLE I. The energy gain of the gravity assist given by the model against the predicted gain given by the theory as a function of approach angle.

Launch Angle (θ)	Closest Approach (m)	Energy Gained (J)	Predicted Energy Gained (J)
30	$3.242E + 08$	$2.963E + 10$	$4.218E + 10$
45	$6.446E + 08$	$2.530E + 10$	$4.170E + 10$
60	$9.749E + 08$	$2.101E + 10$	$4.107E + 10$
90	$1.390E + 09$	$1.497E + 10$	$3.955E + 10$
120	$1.398E + 09$	$1.235E + 10$	$3.803E + 10$
135	$1.398E + 09$	$1.174E + 10$	$3.739E + 10$
150	$1.398E + 09$	$1.138E + 10$	$3.691E + 10$

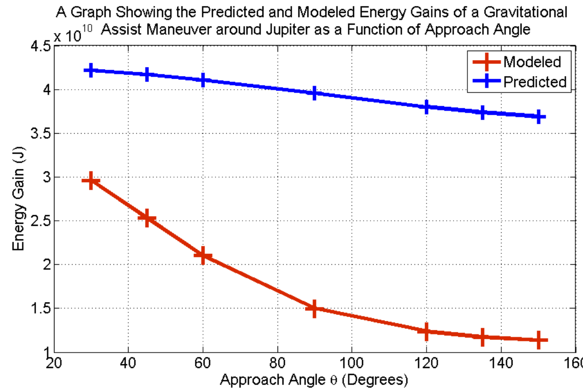

FIG. 3. This graph shows the dependence of the slingshot on the angle between the planet's velocity and the satellites approach velocity, as discussed in appendix B

Appendix C: Matlab Code to Plot a Hyperbolic orbit

This code will plot a hyperbolic orbit around Jupiter for a given set of input parameters.

```
%===============================================
%
%       Title : Analytical 2 Body
%       Date : 20/12/2013
%       Author : Aaron Jones
%
%   Description : This file shows the two body
%               analytical solution for a
%               satellite entering a
%               hyperbollic orbit around
%               Jupiter from infinity. The
%               calculation starts at r = 500
%               radius Jupiter. gamma and v
%               are the control parameters
%
%===============================================
clear all

%Set the angle range to be calculated
ang = (240*pi/180):0.01:(2*pi + (120*pi/180));
% ang = (0):0.01:(2*pi);

%Set the ang=90 radius if the satalite goes in a straight line ratio
%E.g. r=gamma*rj at ang = pi/2 if the satalite goes in a striaght line
%
%----
gamma = 10;
%----


%Set Up Initial Data

rj = 0.6985e8; %Radius Jupiter
ra = gamma.*rj; %As above
rini  = 500*rj; %Initial radius
G = 0.667e-10;  %gravitaional constant
Mj = 0.1903e28; %mass jupiter
Ms = 5;      %mass satalite
%--------
```

```
v = 7.5*sqrt(2*G*Mj/rini);  %Initial velocity = escape velocity
%--------

%Angular and total velocites
vtheta = (v/rini).*ra/rini;
vtot = (sqrt(vtheta.*rini.*vtheta.*rini + v*v));

disp('Set Up Okay');


% Calculate h and lambda

h = abs(rini.*rini.*vtheta); %Total angular momentum

lambda = G*(Mj + Ms);

disp('h and lambda ok');

r0 = h ./ vtot    %Maximum orbit

r1 = 1 ./ (  ((2.*lambda)./(r0.*r0.*vtot.*vtot)) - (1./r0)) %Closest approach


disp ('r1 and r0 okay');

r = 1./ (0.5.*((1./r0) + (1./r1) - ((1./r1) - (1./r0)).*cos(ang))); %Calculate r


%Plot rj to see if there is a collision
size = length(ang);
rj1 = rj*ones(1,size);

%Plot
polar(ang,r);
hold all;
polar(ang,rj1);
hold off
```